

SUBGROUPS AND STRICTLY CLOSED INVARIANT C*-SUBALGEBRAS

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ABSTRACT. We characterise the strictly closed left invariant C*-subalgebras of the C*-algebra $C_b(G)$ of bounded continuous functions on a locally compact group G . On the dual side, we characterise the strictly closed invariant C*-subalgebras of the multiplier algebra of the reduced group C*-algebra $C_r^*(G)$ when G is amenable. In both cases, these C*-subalgebras correspond to closed subgroups of G .

1. INTRODUCTION

Takesaki and Tatsuuma proved in [10] that left translation invariant von Neumann subalgebras of $L^\infty(G)$ for a locally compact group G correspond to closed subgroups H of G . Namely, the left invariant subalgebra is formed by functions in $L^\infty(G)$ that are constant on right cosets of H . Takesaki and Tatsuuma proved also the dual result: invariant von Neumann subalgebras of the group von Neumann algebra $VN(G)$ correspond to closed subgroups of G . In this case, the subalgebra consists of operators in $VN(G)$ supported by the corresponding closed subgroup.

In the C*-algebraic setting, Lau and Losert proved in [7] that left translation invariant C*-subalgebras of C*-algebra $C_0(G)$ of continuous functions on G vanishing at infinity correspond to *compact* subgroups H . Similarly as in the case of $L^\infty(G)$, the subalgebra consists of the functions in $C_0(G)$ that are constants on right cosets of H . The dual version of this result is shown in [9] for amenable locally compact groups: invariant C*-subalgebras of the reduced group C*-algebra $C_r^*(G)$ correspond to *open* subgroups of G , the C*-subalgebra being the collection of elements in $C_r^*(G)$ supported by the open subgroup.

We see that if we look at the C*-algebraic side of things, there is the constraint that the subgroups are either compact or open (which are dual to each other). To go beyond these constraints, we must look into the multiplier algebras. That is, we replace $C_0(G)$ and $C_r^*(G)$ with their multiplier algebras; in the former case this is the C*-algebra $C_b(G)$ of all bounded continuous functions on G . However, these multiplier algebras contain lots of invariant C*-subalgebras that are not associated with subgroups: for example, the C*-subalgebra of weakly almost periodic functions in $C_b(G)$. If we want an association with subgroups, we should consider invariant C*-subalgebras that are closed under the strict topology. Recall that the strict topology on the multiplier algebra $M(A)$ of a C*-algebra A is the topology generated by the seminorms $x \mapsto \|xa\|$, $x \mapsto \|ax\|$, where a runs through the elements of A .

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In this note, we shall prove that there is a one-to-one correspondence between closed subgroups of a locally compact group G and strictly closed left invariant C^* -subalgebras of $C_b(G)$. We shall also prove that for amenable G there is a one-to-one correspondence between closed subgroups and strictly closed invariant C^* -subalgebras of the multiplier algebra of the reduced group C^* -algebra $C_r^*(G)$. These results show that the strict topology, rather than the norm topology, is often the right topology for C^* -algebras.

2. STRICTLY CLOSED LEFT INVARIANT C^* -SUBALGEBRAS OF BOUNDED CONTINUOUS FUNCTIONS

In this section we characterise the strictly closed left invariant C^* -subalgebras of $C_b(G)$ where G is a locally compact group. These correspond to the closed subgroups of G . The proof follows along the same lines as that of [7, Lemma 12], but of course the use of strict topology requires some subtlety.

We let L_s and R_s denote the left and right translation operators: $L_s f(t) = f(st)$, $R_s f(t) = f(ts)$ where $s, t \in G$ and $f: G \rightarrow \mathbb{C}$.

Theorem 1. *Let G be a locally compact group. There is a one-to-one correspondence between closed subgroups H of G and strictly closed, left invariant C^* -subalgebras X of $C_b(G)$:*

- (1) $X = \{ f \in C_b(G); R_s f = f \text{ for every } s \in H \}$
- (2) $H = \{ s \in G; R_s f = f \text{ for every } f \in X \}.$

Moreover, H is normal if and only if the corresponding X is right invariant.

Proof. Suppose that H is a closed subgroup of G and let X be defined by (1). Then X is obviously a left invariant C^* -subalgebra of $C_b(G)$. Suppose f is in the strict closure of X in $C_b(G)$. Since strict convergence implies pointwise convergence, it follows that also f satisfies $R_s f = f$. So X is strictly closed.

Conversely, suppose that a strictly closed, left invariant C^* -subalgebra X is given and define H by (2). Then H is a closed subgroup of G . Let G/H denote the homogeneous space of right cosets of H and define

$$\pi: C_b(G/H) \rightarrow C_b(G), \quad \pi(f)(s) = f(sH).$$

Using the fact that on bounded sets the strict topology agrees with the compact-open topology [1], it is easy to see that π is strictly continuous on bounded sets. Taylor has shown in [12, Corollary 2.7] that the strongest locally convex topology agreeing with the strict topology on norm-bounded sets is actually the strict topology itself. Consequently, a linear map from a multiplier algebra to a locally convex space is strictly continuous if it is strictly continuous on bounded sets. Therefore, π is in fact strictly continuous. Moreover, π is a $*$ -isomorphism from $C_b(G/H)$ onto

$$Y := \{ f \in C_b(G); R_s f = f \text{ for every } s \in H \}.$$

Obviously $X \subseteq Y$. To show that $X = Y$, we shall apply the strict Stone–Weierstrass theorem due to Glicksberg [5]: if a strictly closed C^* -subalgebra X of $C_b(\Omega)$ separates points of the locally compact space Ω , then $X = C_b(\Omega)$.

First of all, $\pi^{-1}(X)$ is a strictly closed C^* -subalgebra of $C_b(G/H)$ (because π is strictly continuous). To show that $\pi^{-1}(X)$ separates points of G/H , let $s, t \in G$ with $s^{-1}t \notin H$. By the definition of H , there is f in X such that $f(us^{-1}t) \neq f(u)$ for some u in G . Using the left invariance of X , we have $f(s) \neq f(t)$ and

so $\pi^{-1}(f)(sH) \neq \pi^{-1}(f)(tH)$. Then it follows from the strict Stone–Weierstrass theorem that $X = Y$.

This argument also shows that if we start from X and define H by (2), then X_H derived from H by (1) is equal to X . On the other hand, if we start from H and define X by (1), then X is $*$ -isomorphic to $C_b(G/H)$. Now if $s \notin H$, then there is f in $C_b(G/H)$ such that $f(sH) \neq f(H)$. Taking in count the identification of X with $C_b(G/H)$, this means that s is not in the subgroup H_X arising from X via (2). That is, $H = H_X$.

It remains to show that H is normal if and only if the corresponding X is right invariant. Now X is right invariant if and only if $R_h R_s f = R_s f$ for every $f \in X$, $s \in G$ and $h \in H$. But $R_h R_s f = R_s R_{s^{-1}hs} f$ so X is right invariant if and only if $R_{s^{-1}hs} f = f$ for every $f \in X$, $s \in G$ and $h \in H$. The last condition is equivalent with normality of H . \square

3. STRICTLY CLOSED INVARIANT C*-SUBALGEBRAS OF THE MULTIPLIER ALGEBRA OF THE GROUP C*-ALGEBRA

Let G be an amenable locally compact group, and let λ be the left regular representation of G on $L^2(G)$. We shall consider the reduced group C*-algebra $C_r^*(G)$ generated by $\lambda(L^1(G))$. Of course since we take G amenable, the reduced group C*-algebra of G is isomorphic with the universal group C*-algebra. The dual space of $C_r^*(G)$ is the reduced Fourier–Stieltjes algebra $B_r(G)$, which is a Banach algebra under pointwise multiplication of functions (Eymard introduced the reduced Fourier–Stieltjes algebra in [4], denoting it by $B_\rho(G)$). In the amenable case, $B_r(G)$ coincides with the Fourier–Stieltjes algebra $B(G)$ consisting of coefficients of all unitary representations of G .

We shall also need the group von Neumann algebra $VN(G)$, which is the von Neumann algebra generated by λ . The predual of $VN(G)$ is the Fourier algebra $A(G)$ consisting of the coefficient functions of λ . The multiplier algebra $M(C_r^*(G))$ is identified with the idealiser of $C_r^*(G)$ in $VN(G)$:

$$M(C_r^*(G)) = \{x \in VN(G); xa, ax \in C_r^*(G) \text{ for every } a \in C_r^*(G)\}.$$

We note that elements in $B_r(G)$ have unique strictly continuous extensions to functionals on $M(C_r^*(G))$, and we shall use these extensions without additional notation.

Define a unitary operator W on $L^2(G \times G)$ by

$$W\xi(s, t) = \xi(ts, t) \quad (\xi \in L^2(G \times G), s, t \in G).$$

Then

$$\Gamma: x \mapsto W^*(1 \otimes x)W \quad (x \in M(C_r^*(G)))$$

maps $M(C_r^*(G))$ to $M(C_r^*(G) \otimes C_r^*(G)) \subseteq B(L^2(G \times G))$ (e.g. $\Gamma(\lambda(s)) = \lambda(s) \otimes \lambda(s)$ for $s \in G$). We have an action of $B_r(G)$ on $M(C_r^*(G))$ via

$$u.x = (u \otimes \text{id})\Gamma(x) \quad (u \in B_r(G), x \in M(C_r^*(G)))$$

where $u \otimes \text{id}$ denotes the unique strictly continuous extension of the linear map from $C_r^*(G) \otimes C_r^*(G)$ to $C_r^*(G)$ determined by $a \otimes b \mapsto u(a)b$. We say that a C*-subalgebra X of $M(C_r^*(G))$ is *invariant* if $u.x \in X$ for every $x \in X$ and $u \in B_r(G)$. To see that this agrees with the notion of invariance used in $C_b(G)$ (i.e. translation invariance), note that

$$\langle u.x, v \rangle = \langle x, uv \rangle \quad (x \in M(C_r^*(G)), u, v \in B_r(G)).$$

Now if G is abelian, $M(C_r^*(G)) \cong C_b(\widehat{G})$ and $X \subseteq C_b(\widehat{G})$ is invariant if and only if it is translation invariant (if $u = \chi$ is a character and x corresponds to a continuous function $f \in C_b(\widehat{G})$, then $u.x$ corresponds to the translation of f by χ).

Since G is amenable, there exists a *summing net* $\{F_\alpha\}$ that satisfies the following properties

- each F_α is nonnull and compact
- $F_\alpha \subseteq F_\beta$ if $\alpha \leq \beta$
- $G = \bigcup_\alpha F_\alpha$
- $|sF_\alpha \triangle F_\alpha|/|F_\alpha| \rightarrow 0$ uniformly on compact sets

(see [8, Theorem 4.16]). Here we use $|\cdot|$ to denote the left Haar measure of a set and \triangle to denote the symmetric difference of sets.

Put $\zeta_\alpha = |F_\alpha|^{-1/2} 1_{F_\alpha} \in L^2(G)$ and $u_\alpha = \zeta_\alpha * \check{\zeta}_\alpha$, where $\check{\zeta}_\alpha(s) = \zeta_\alpha(s^{-1})$, so that $\langle u_\alpha, x \rangle = (x\zeta_\alpha | \zeta_\alpha)$ for every $x \in C_r^*(G)$. Then each u_α is compactly supported and (u_α) is a bounded approximate identity in $A(G)$.

Lemma 2. *For every $x \in M(C_r^*(G))$, $u_\alpha.x \rightarrow x$ in the strict topology.*

Proof. We should show that

$$(u_\alpha.x)a \rightarrow xa \quad \text{and} \quad a(u_\alpha.x) \rightarrow ax$$

in norm for every a in $C_r^*(G)$. Consider the first case. Now

$$\|(u_\alpha.x)a - xa\| \leq \|(u_\alpha.x)a - u_\alpha.(xa)\| + \|u_\alpha.(xa) - xa\|,$$

and since $xa \in C_r^*(G)$

$$\|u_\alpha.(xa) - xa\| \rightarrow 0.$$

So we are left to show that

$$\|(u_\alpha.x)a - u_\alpha.(xa)\| \rightarrow 0.$$

Now suppose first that $a = \lambda(f)$ for some f in $L^1(G)$ with compact support K . Given $\epsilon > 0$, choose α_0 such that

$$\frac{|sF_\alpha \triangle F_\alpha|}{|F_\alpha|} < \epsilon$$

for every s in K and $\alpha \geq \alpha_0$.

Now every $v \in A(G)$ is of the form $v = \bar{\xi} * \check{\eta}$ where $\xi, \eta \in L^2(G)$ are such that $\|v\| = \|\xi\|_2 \|\eta\|_2$. Then

$$\begin{aligned} & |\langle (u_\alpha.x)\lambda(f) - u_\alpha.(x\lambda(f)), v \rangle| \\ &= |(W^*(1 \otimes x)W(1 \otimes \lambda(f))(\zeta_\alpha \otimes \eta) - W^*(1 \otimes x\lambda(f))W(\zeta_\alpha \otimes \eta) | \zeta_\alpha \otimes \xi)| \\ &= |(W(1 \otimes \lambda(f))(\zeta_\alpha \otimes \eta) - (1 \otimes \lambda(f))W(\zeta_\alpha \otimes \eta) | (1 \otimes x^*)W(\zeta_\alpha \otimes \xi))| \\ &\leq \|W(1 \otimes \lambda(f))(\zeta_\alpha \otimes \eta) - (1 \otimes \lambda(f))W(\zeta_\alpha \otimes \eta)\|_2 \|x\| \|\xi\|_2. \end{aligned}$$

Now

$$\begin{aligned} & \|W(1 \otimes \lambda(f))(\zeta_\alpha \otimes \eta) - (1 \otimes \lambda(f))W(\zeta_\alpha \otimes \eta)\|_2 \\ &= \left(\iint \left| \int f(u) \zeta_\alpha(ts) \eta(u^{-1}t) - f(u) \zeta_\alpha(u^{-1}ts) \eta(u^{-1}t) du \right|^2 ds dt \right)^{1/2} \\ &\leq \left(\iint \left(\int |f(u)| |\zeta_\alpha(ts) - \zeta_\alpha(u^{-1}ts)| |\eta(u^{-1}t)| du \right)^2 ds dt \right)^{1/2}. \end{aligned}$$

Plug in

$$|\zeta_\alpha(ts) - \zeta_\alpha(u^{-1}ts)| = \frac{1_{uF_\alpha \triangle F_\alpha}(ts)}{|F_\alpha|^{1/2}},$$

and apply Minkowski's integral inequality. Then we have

$$\begin{aligned} & \|W(1 \otimes \lambda(f))(\zeta_\alpha \otimes \eta) - (1 \otimes \lambda(f))W(\zeta_\alpha \otimes \eta)\|_2 \\ & \leq \int \left(\iint |f(u)|^2 \frac{1_{uF_\alpha \triangle F_\alpha}(ts)}{|F_\alpha|} |\eta(u^{-1}t)|^2 ds dt \right)^{1/2} du \\ & = \|\eta\|_2 \int |f(u)| \frac{|uF_\alpha \triangle F_\alpha|^{1/2}}{|F_\alpha|^{1/2}} du < \epsilon^{1/2} \|\eta\|_2 \|f\|_1 \end{aligned}$$

for every $\alpha \geq \alpha_0$. Therefore

$$|\langle (u_\alpha \cdot x)\lambda(f) - u_\alpha \cdot (x\lambda(f)), v \rangle| < \epsilon^{1/2} \|f\|_1 \|x\| \|v\|$$

for every $\alpha \geq \alpha_0$. It follows that

$$\|(u_\alpha \cdot x)\lambda(f) - u_\alpha \cdot (x\lambda(f))\| \rightarrow 0$$

and by approximation

$$\|(u_\alpha \cdot x)a - u_\alpha \cdot (xa)\| \rightarrow 0,$$

as required.

That also $a(u_\alpha \cdot x) \rightarrow ax$ can be proved similarly. \square

The support $\text{supp } x$ of an operator $x \in \text{VN}(G)$ is defined as follows: $s \in G$ is in $\text{supp } x$ if and only if for every neighbourhood U of s there is $v \in A(G)$ supported by U such that $\langle x, v \rangle \neq 0$.

Lemma 3. *Let G be an amenable locally compact group and H a closed subgroup of G . Then*

$$\{x \in M(C_r^*(G)); \text{supp } x \subseteq H\} = \overline{\text{span}} \lambda(H)$$

where the $\overline{\text{span}}$ denotes the strictly closed linear span.

Proof. It follows from Proposition 4.8 of [4] that the set on the left-hand side of the identity is strictly closed. Since $\text{supp } \lambda(h) = \{h\}$ for every h in H , we see that $\overline{\text{span}} \lambda(H)$ is contained in the set on the left-hand side of the identity.

Conversely, let $x \in M(C_r^*(G))$ be supported by H . Suppose first that x is compactly supported. Since x is in the double commutant $\lambda(H)''$, there is a bounded net $(x_\alpha)_{\alpha \in I} \subseteq \text{span } \lambda(H)$ such that $x_\alpha \rightarrow x$ in the weak* topology. Since x is compactly supported, we may assume without loss of generality that there is a compact set K that is a common support for x and for each x_α (otherwise we replace x_α with $u \cdot x_\alpha$ where u is a compactly supported function in $A(G)$ with $u = 1$ on a neighbourhood of the support of x). Let $y \in C_r^*(G)$ be compactly supported. Then $\text{supp}(xy) \subseteq K \text{supp } y$ and $\text{supp}(x_\alpha y) \subseteq K \text{supp } y$ (by [4, Proposition 4.8]). Now there is u in $A(G)$ such that $u = 1$ on a neighbourhood of the compact set $K \text{supp } y$. Then, for every v in $B_r(G)$, the function uv is in $A(G)$ and so

$$\langle x_\alpha y, v \rangle = \langle u \cdot (x_\alpha y), v \rangle = \langle x_\alpha y, uv \rangle \rightarrow \langle xy, uv \rangle = \langle xy, v \rangle.$$

Therefore $x_\alpha y \rightarrow xy$ weakly in $C_r^*(G)$ for every compactly supported y in $C_r^*(G)$. Since the net (x_α) is bounded, we get rid of the restriction of compact support by approximation. We can deal with the other side similarly, so for every y in $C_r^*(G)$

$$x_\alpha y \rightarrow xy \text{ weakly and } yx_\alpha \rightarrow yx \text{ weakly.}$$

We get from weak convergence to norm convergence by a typical convexity argument such as the one presented in [3, p. 524]. Since there is the subtlety to obtain a *single* net (z_β) of convex combinations of x_α 's that works for *every* y in $C_r^*(G)$, we repeat here the argument of Day [3].

We say that (x'_β) is a net of convex combinations far out in (x_α) if for every α_0 there is β_0 such that each x'_β with $\beta \geq \beta_0$ is a convex combination of x_α 's with $\alpha \geq \alpha_0$. Let $F = \{y_1, y_2, \dots, y_n\}$ be a finite subset of $C_r^*(G)$. It follows from the Hahn–Banach theorem that there is a net (x_β^1) of convex combinations far out in (x_α) such that $x_\beta^1 y_1 \rightarrow xy_1$ in norm. Now $x_\beta^1 y_2 \rightarrow xy_2$ weakly since $(x_\beta^1 y_2)$ is a net of convex combinations far out in $(x_\alpha y_2)$. Hence there is a net (x_γ^2) of convex combinations far out in (x_β^1) such that $x_\gamma^2 y_j \rightarrow xy_j$ in norm for $j = 1, 2$. Inductively, we get a net (x_β^F) of convex combinations far out in (x_α) such that $x_\beta^F y \rightarrow xy$ in norm for every $y \in F$.

Let \mathcal{F} be the collection of all finite subsets of $C_r^*(G)$, and recall that we denote the index set of α 's by I . Then $\mathcal{F} \times I$ is a directed set where $(F_1, \alpha_1) \geq (F_2, \alpha_2)$ if and only if $F_1 \supseteq F_2$ and $\alpha_1 \geq \alpha_2$. Now for every $(F, \alpha_0) \in \mathcal{F} \times I$ pick $z(F, \alpha_0)$ such that $z(F, \alpha_0)$ is a convex combination of elements x_α where $\alpha \geq \alpha_0$ and

$$\|z(F, \alpha_0)y - xy\| < \frac{1}{|F|}$$

for every $y \in F$ (here $|F|$ denotes the cardinality of F). This choice is possible by the construction above. Then for every y in $C_r^*(G)$, the net $(z(F, \alpha)y)_{(F, \alpha) \in \mathcal{F} \times I}$ converges to xy in norm. Since $(z(F, \alpha))$ is a net of convex combinations far out in (x_α) , we have $yz(F, \alpha) \rightarrow yx$ weakly for every $y \in C_r^*(G)$, so we may repeat the argument to obtain a net (z_β) of convex hull of (x_α) such that $z_\beta \rightarrow x$ in the strict topology. This completes the case when x is compactly supported.

To deal with the case when the support of x is not necessarily compact, we need amenability. Let (u_α) be the bounded approximate identity from Lemma 2. Since each u_α is compactly supported, so is $u_\alpha x$. Moreover, since x is supported by H , so is each $u_\alpha x$. By Lemma 2, $u_\alpha x \rightarrow x$ strictly so we may apply the earlier part of the proof to $u_\alpha x$'s and obtain that $x \in \overline{\text{span}} \lambda(H)$. \square

Theorem 4. *Let G be an amenable locally compact group. There is a one-to-one correspondence between closed subgroups H of G and strictly closed, invariant C^* -subalgebras X of $M(C_r^*(G))$:*

$$(3) \quad X = \{x \in M(C_r^*(G)); \text{supp } x \subseteq H\}$$

$$(4) \quad H = \{s \in G; \lambda(s) \in X\}.$$

Proof. If H is given, then (3) defines a strictly closed, invariant C^* -subalgebra X of $M(C_r^*(G))$. Moreover, $X = \overline{\text{span}} \lambda(H)$ by Lemma 3, which implies that the application of (4) rediscovers H .

Conversely, let X be a strictly closed, invariant C^* -subalgebra of $M(C_r^*(G))$, and define H by (4). Then H is obviously a subgroup and it is not difficult to see that H is closed. Indeed, if $f \in L^1(G)$, then $s \mapsto L_s f$ and $s \mapsto R_s f$ are continuous maps from G to $L^1(G)$. Therefore $s \mapsto \lambda(s)\lambda(f)$ and $s \mapsto \lambda(f)\lambda(s)$ are continuous, and hence $s \mapsto \lambda(s)$ is strictly continuous. Since X is strictly closed, it follows that H is closed.

Put

$$Y = \{x \in M(C_r^*(G)); \text{supp } x \subseteq H\}.$$

It follows from Lemma 3 that $Y \subseteq X$. On the other hand, let $x \in X$ such that $\text{supp } x$ is compact. For every $s \in \text{supp } x$, there is a net (v_α) of functions in $A(G)$ such that $v_\alpha \cdot x \rightarrow \lambda(s)$ in the weak* topology ([4]). Now $v_\alpha \cdot x \in X$ by invariance. The technique used in the proof of Lemma 3 allows us to pass from weak* topology to strict topology, and so it follows that $\lambda(s)$ is in X . Therefore $\text{supp } x \subseteq H$; that is, $x \in Y$. The general case, when $\text{supp } x$ is not necessarily compact, is covered by using the bounded approximate identity in $A(G)$ from Lemma 2. Hence $Y = X$, which also shows that passing from X to H and then applying (3) takes us back to X . \square

It seems reasonable to conjecture that given a closed subgroup H of an amenable locally compact group G , the C*-algebra

$$X = \{ x \in M(C_r^*(G)); \text{supp } x \subseteq H \}$$

is canonically *-isomorphic to $M(C_r^*(H))$. We end this note with some comments on this conjecture.

First of all, let us embed $M(C_r^*(H))$ into $M(C_r^*(G))$. There is a normal injective *-homomorphism $\pi: VN(H) \rightarrow VN(G)$ which maps each $\lambda_H(h)$ to $\lambda_G(h)$ (see [11] and [6]). Restricted to $M(C_r^*(H))$ this map is a strictly continuous injective *-homomorphism. (Indeed, π defines a unitary representation of H on $L^2(G)$, which leads to a nondegenerate, faithful representation of $C_r^*(H) = C^*(H)$ on $L^2(G)$. Due to nondegeneracy, its strict extension from $M(C_r^*(H))$ into $M(C_r^*(G))$ agrees with the normal *-homomorphism π that we started with.)

So we have an embedding $\pi: M(C_r^*(H)) \rightarrow M(C_r^*(G))$. Since the image of π is strictly dense in X by Lemma 3, for $\pi: M(C_r^*(H)) \rightarrow X$ to be a *-isomorphism, it suffices that the range of π is strictly closed. Whether that is the case is unknown (to me). However, the stronger statement that π is a strict homeomorphism onto its range is certainly false. To see this, suppose that π is a strict homeomorphism onto its range. Then, by the Hahn–Banach theorem, every strictly continuous functional on $\pi(M(C_r^*(H)))$ extends to a strictly continuous functional on $M(C_r^*(G))$. That is, for every u in $B(H)$, there is v in $B(G)$ such that $u = v \circ \pi$ as functionals. Evaluating at $\lambda(h)$ says that $u(h) = v(h)$. Therefore every function in $B(H)$ extends to a $B(G)$ function. It is known that this is not true in general, for example for the affine group on the real line [4, 2].

The problem whether the range of π is strictly closed can be reduced to the elements outside $UCB(\widehat{G})$, the norm closure of compactly supported elements in $VN(G)$. Suppose that (x_α) is a net in $\pi(M(C_r^*(H)))$ converging strictly to x in $M(C_r^*(G))$. If x has compact support, then an argument similar to the proof of Lemma 3 shows that x is in fact in the range of π . Approximation in norm extends this observation to all $x \in UCB(\widehat{G})$.

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REFERENCES

- [1] R. C. Buck, *Bounded continuous functions on a locally compact space*, Michigan Math. J. **5** (1958), 95–104.
- [2] L. N. Carling, *On the restriction map of the Fourier-Stieltjes algebra $B(G)$ and $B_\rho(G)$* , J. Funct. Anal. **25** (1977), 236–243.
- [3] M. M. Day, *Amenable semigroups*, Illinois J. Math. **1** (1957), 509–544.
- [4] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France **92** (1964), 181–236.
- [5] I. Glicksberg, *Bishop's generalized Stone-Weierstrass theorem for the strict topology.*, Proc. Amer. Math. Soc. **14** (1963), 329–333.
- [6] C. S. Herz, *Harmonic synthesis for subgroups*, Ann. Inst. Fourier **23** (1973), 91–123.
- [7] A. T.-M. Lau and V. Losert, *Complementation of certain subspaces of $L_\infty(G)$ of a locally compact group*, Pacific J. Math. **141** (1990), 295–310.
- [8] A. L. T. Paterson, *Amenability*, Mathematical Surveys and Monographs, vol. 29, American Mathematical Society, Providence, RI, 1988.
- [9] P. Salmi, *Compact subgroups and left invariant C^* -subalgebras of locally compact quantum groups*, J. Funct. Anal. **261** (2011), 1–24.
- [10] M. Takesaki and N. Tatsuuma, *Duality and subgroups*, Ann. of Math. (2) **93** (1971), 344–364.
- [11] ———, *Duality and subgroups. II*, J. Funct. Anal. **11** (1972), 184–190.
- [12] D. C. Taylor, *The strict topology for double centralizer algebras*, Trans. Amer. Math. Soc. **150** (1970), 633–643.

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